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On zeros of discrete orthogonal polynomials

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Abstract

We exploit difference equations to establish sharp inequalities on the extreme zeros of the classical discrete orthogonal polynomials, Charlier, Krawtchouk, Meixner and Hahn. We also provide lower bounds on the minimal distance between their consecutive zeros.

Keywords: Discrete orthogonal polynomials; Charlier polynomials; Krawtchouk polynomials; Meixner polynomials; Hahn polynomials; Zeros

1. Introduction

The aim of this paper is to establish sharp inequalities for the extreme zeros of the classical discrete orthogonal polynomials, namely, Charlier, Krawtchouk, Meixner and Hahn. Our approach is based on the corresponding difference equations and it also provides some information about the spacing of the zeros. In particular, we derive a lower bound on the minimum distance between consecutive zeros of Charlier, Krawtchouk, Meixner and symmetric Hahn polynomials. A similar technique applied earlier in the continuous case gave currently best known inequalities for the zeros of Laguerre and Jacobi polynomials [7]. Moreover, it is known that in the continuous case the bounds are essentially sharp up to a constant factor at the second order term [9]. By analogy, we believe that the inequalities obtained in this paper are sharp in the same sense, although we have no rigorous proof.

An example of second order bounds is provided by the classical inequality of Szegö [15] stating that the largest zero of the Hermite polynomial $H_k(x)$ does not exceed $\sqrt{2k} + ck^{-1/6}$,

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where c is an explicitly known constant, whereas the first order bound is just $\sqrt{2k}$. In this particular case the second order term is sharp in the asymptotic sense. In general, as we are looking for inequalities which hold for a wide range of parameters including the degree k, we have to allow a weaker constant c than the asymptotically sharp one.

In the discrete case the only previously known results going beyond the first order bounds were obtained for the binary Krawtchouk and the Charlier polynomials [5,6]. Here we improve slightly these inequalities and establish similar results for the rest of the cases. For a half-infinite interval of orthogonality our bounds for the extreme zeros take the form $c_1 k \pm c_2 k^{1/3}$, provided the bounded parameters are fixed and the unbounded parameters are linear in the degree. Here $c_1, c_2 > 0$, are explicitly written constants, $c_1 k$ is the first order bound and the plus sign corresponds to the least zero. For a finite interval $[0, n]$, under similar conditions, our bounds look as $c_1 n \pm c_2 n^{1/3}$.

It is worth noticing that to find the first order bounds is much easier, provided the corresponding three term recurrence is known, and the task may be accomplished in many ways. One of the simplest possibilities is to apply Gershgorin's theorem to the Jacobi matrix arising from the three term recurrence. A slight improvement, yet giving only first order bounds, can be obtained via chain sequences technique, see e.g. [3] and the references therein. Another approach is discussed in [10,11]. For symmetric orthogonal polynomials sharp second order asymptotics and inequalities are known in terms of the three term recurrence relation [8,13]. When applied to the symmetric Hahn and Krawtchouk polynomials they yield bounds very similar, in fact, different only by the constant before the second order term, to those obtained in this paper.

Our aim here is more modest. Although we deduce our results from a general statement we will deal mainly with special functions rather than with orthogonal polynomials per se. Nevertheless it is worth noticing a recent result stating that polynomials orthogonal on a finite or half-infinite set of consecutive integers satisfy a second order linear difference equation [2,3].

To avoid unnecessary discussion of trivial situations we will assume in what follows that all orthogonal polynomials have degree at least two. We refer to [3,14] for the basic formulas used in what follows.

2. General results

Discrete orthogonal polynomials are orthogonal polynomials with the corresponding measure supported on a subset $L \subset \mathbb{Z}$. That is the orthogonality relation can be written as

$$\sum_{x \in L} w(x) p_i(x) p_j(x) = \delta_{ij} \| p_i \|^2.$$

In the classical case $L = \{0, 1, \ldots, n\}$ for Krawtchouk and Hahn, and $L = \{0, 1, \ldots\}$ for Charlier and Meixner polynomials, respectively.

Our approach does not appeal directly to orthogonality and is based on the fact that we deal with hyperbolic polynomials, that is with real polynomials with only real zeros. Let $p = p(x)$ be such a polynomial, the mesh $\mathcal{M}(p)$ is defined as the minimum distance between its zeros. The necessity to cope with this notion is the main difference between the continuous and the discrete cases.

We assume that for a given degree k, a discrete orthogonal polynomial $p = p_k(x)$ with the zeros $x_1 < x_2 < \cdots < x_k$, satisfies the difference equation

$$p(x+1) = 2A(x) p(x) - B(x) p(x-1). \tag{1}$$

It will be convenient to denote by x_0 and x_{k+1} the least and the largest (possibly infinite) points of L and set $\mathcal{L} = (x_0, x_{k+1})$, $\bar{\mathcal{L}} = [x_0, x_{k+1}]$. We also assume that $A(x)$ and $B(x)$ are continuous functions on $\mathcal{L}$.

It seems that the following easy to prove general result has been noticed only recently [5], although the corresponding statements for binary Krawtchouk and Hahn polynomials were known for a long time [1,12].

Theorem 1. *Let $p(x)$ be a non-identically zero discrete orthogonal polynomial, corresponding to an orthogonality measure supported on a subset of integers. Suppose that $p(x)$ satisfies (1) and there exists an open interval $I \subset \mathcal{L}$ such that*

(i) *All zeros of p lie in I,*
(ii) $B(x) > 0$ *for $x \in I$.*

Then $\mathcal{M}(p) > 1$. If, in addition, $A(x) > 0$ on I then $\mathcal{M}(p) > 2$.

Proof. Let $z_1 < z_2$ be the largest pair of the consecutive zeros of $p = p_k$ satisfying $z_2 - z_1 < 1$. Since there exists a point of the support of the measure between any two zeros of p we conclude that there is a unique integer $m \in L$ such that $z_1 \le m \le z_2$. Moreover, for any zero $z_3 < z_1$ we have $z_3 \le m - 1$. Assume first that $m \ne z_1, z_2$. Then

$$sgn\ p(z_2 + 1) = sgn\ p(m + 1) = -sgn\ p(m),$$

and

$$sgn\ p(z_2 - 1) = sgn\ p(m - 1) = -sgn\ p(m),$$

contradicting $p(z_2 + 1) = -B_k(z_2)p(z_2 - 1)$ and $B_k(z_2) > 0$. The cases $m = z_1$ or z_2 are similar. Thus, we obtain $\mathcal{M}(p) \ge 1$. The case $\mathcal{M}(p) = 1$ is impossible as then $p(z_2 + 1) = 0$ and therefore $p(z_2 + i) = 0$ for any non-negative integer i.

Now, assuming $A(x) > 0$ consider two consecutive zeros $z_1 < z_2$ such that $z_2 - z_1 < 2$. Setting $z = (z_1 + z_2)/2$ we have

$$sgn\ p(z - 1) = -sgn\ p(z) = sgn\ p(z + 1),$$

contradicting (1). The case $\mathcal{M}(p) = 2$ is impossible since otherwise $A(z)p(z) = 0$ implying $p(z) = 0$. $\square$

It is worth noticing that the condition $B(x) > 0$ is fulfilled for all classical discrete orthogonal polynomials besides a small range of the parameters in the Hahn case.

We need some more notation to state our results. We define the following three functions playing an important role in what follows.

$$t(x) = \frac{p(x-1)}{p(x)},$$
$$t_0(x) = \frac{A(x)}{B(x)},$$
$$\Delta(x) = B(x) - A^2(x).$$

Let $\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_k$ be the branches of $t(x)$, where $\mathcal{B}_i$ is defined on the interval (x_i, x_{i+1}), $i = 0, 1, \ldots, k$. By y_i we denote the intersection point of $\mathcal{B}_i$ and $t_0(x)$ on $\mathcal{L}$, if such a point exists.

We will use the following obvious product representation

$$\frac{t(x)}{t(x+1)} = \frac{p(x-1)p(x+1)}{p^2(x)} = \prod_{i=1}^{k}\left(1 - \frac{1}{(x-x_i)^2}\right).$$

In view of (1), it may be rewritten as

$$2A(x)t(x) - B(x)t^2(x) = \prod_{i=1}^{k}\left(1 - \frac{1}{(x-x_i)^2}\right). \tag{2}$$

Notice also that if $\mathcal{M}(p) \geq 1$, then at most one or two consecutive factors in the above product maybe negative, depending on whether $x \notin [x_1, x_k]$ or $x \in (x_i, x_{i+1})$, for some $i = 1, \ldots, k-1$.

Theorem 2. *Suppose that $B(x) > 0$ for $x \in \mathcal{L}$. Suppose also that the equation $\Delta(x) = 0$ has precisely two zeros $\mu_1 < \mu_2$ in $\bar{\mathcal{L}}$, such that $\Delta(x) > 0$ for $x \in (\mu_1, \mu_2)$. Then the following first order bounds hold:*

$$x_1 > \mu_1, \tag{3}$$

provided y_0 exists;

$$x_k < \mu_2, \tag{4}$$

provided y_k exists.

Moreover, let ν_1, ν_2 be any numbers such that $x_1 \leq \nu_1 \leq \mu_2$, $\mu_1 \leq \nu_2 \leq x_2$. Then

$$x_1 > \min_{x \in (\mu_1, \nu_1)} \left(x + \sqrt{\frac{B(x)}{\Delta(x)}}\right), \tag{5}$$

provided y_0 exists;

$$x_k < \max_{x \in (\nu_2, \mu_2)} \left(x - \sqrt{\frac{B(x)}{\Delta(x)}}\right), \tag{6}$$

provided y_k exists.

If $t_0(x)$ intersects $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$ and $\mathcal{M}(p) \geq 2$, then

$$x_{i+1} - x_i \geq 2 \min_{x_i < x < x_{i+1}} \frac{B^{1/4}(x)}{\sqrt{\sqrt{B(x)} - A(x)}}. \tag{7}$$

In particular,

$$\mathcal{M}(p) > 2 \min_{\mu_1 < x < \mu_2} \frac{B^{1/4}(x)}{\sqrt{\sqrt{B(x)} - A(x)}} \tag{8}$$

provided y_0 and y_k exist and $A(x) > 0$ for $\mu_1 < x < \mu_2$.

Proof. First we observe that $y_0 < x_1, \quad x_k < y_k$. For $x = y_0$ we have

$$2A(x)t(x) - B(x)t^2(x) = 2A(x)t_0(x) - B(x)t_0^2(x) = \frac{A^2(y_0)}{B(y_0)} \geq 0,$$

by the assumption $B(y_0) > 0$. If $A(y_0) = 0$ then $\Delta(y_0) = B(y_0) > 0$. Otherwise all factors on the right hand side of (2) must be positive. Hence

$$\frac{A^2(y_0)}{B(y_0)} < 1 - \frac{1}{(x_1 - y_0)^2}. \tag{9}$$

Notice that $\frac{A^2(y_0)}{B(y_0)} < 1$ means $\Delta(y_0) > 0$. In the same way we get $\Delta(y_k) > 0$. This proves (3) and (4).

Next, we find by (9)

$$x_1 > y_0 + \sqrt{\frac{B(y_0)}{\Delta(y_0)}} \geq \min_{x \in (\mu_1, \nu_1)} \left(x + \sqrt{\frac{B(x)}{\Delta(x)}} \right),$$

giving (5). The proof of (6) is similar.

To demonstrate (7) we choose $x = y_i$ obtaining

$$\begin{aligned}\frac{A^2(y_i)}{B(y_i)} &= \left(1 - \frac{1}{(y_i - x_i)^2}\right)\left(1 - \frac{1}{(x_{i+1} - y_i)^2}\right) \prod_{j \neq i, i+1} \left(1 - \frac{1}{(y_i - x_j)^2}\right) \\ &\leq \left(1 - \frac{1}{(y_i - x_i)^2}\right)\left(1 - \frac{1}{(x_{i+1} - y_i)^2}\right) \\ &\leq \max_{x_i < x < x_{i+1}} \left(1 - \frac{1}{(x - x_i)^2}\right)\left(1 - \frac{1}{(x - x_{i+1})^2}\right).\end{aligned}$$

It is an easy exercise to verify that for $x_{i+1} - x_i > \frac{2}{\sqrt{3}}$ the maximum here is attained for $x = \frac{x_i + x_{i+1}}{2}$ yielding

$$\frac{A^2(y_i)}{B(y_i)} \leq \left(1 - \frac{4}{(x_{i+1} - x_i)^2}\right)^2.$$

Moreover, for $x_{i+1} - x_i > 2$ we also have

$$1 - \frac{4}{(x_{i+1} - x_i)^2} > 0,$$

implying (7).

Finally, (8) immediately follows from (7) and Theorem 1. □

It would be important to find the corresponding inequalities in the opposite direction. In particular, in most of the cases it is difficult to minimize (7) without an appropriate upper bound. We conjecture that for the classical discrete orthogonal polynomials the following is true:

Conjecture 1. *Inequalities (5) and (6) are sharp up to a numerical factor before the second order term.*

Conjecture 2. *There is an absolute constant c such that*

$$x_{i+1} - x_i < c \min_{x_i < x < x_{i+1}} \frac{B^{1/4}(x)}{\sqrt{\sqrt{B(x)} - A(x)}}, \quad i = 1, 2, \ldots, k - 1.$$

Notice that a trivial upper bound on the mesh is given by the average spacing of the zeros,

$$\mathcal{M}(p) < \frac{\mu_2 - \mu_1}{k-1}.$$

Remark 1. Finding the extrema of the function in Theorem 2 can be rather technically involved. Therefore we would like to indicate what types of bounds one can expect to obtain from (5) and (6) for sufficiently smooth functions A and B. For the classical discrete orthogonal polynomials A and B are rational functions in all the variables. Thus, $\frac{B(x)}{\Delta(x)} = \frac{P(x)}{Q(x)}$, where P and Q are coprime polynomials. One may expect that the extremal value $x = x_1^*$ in (5) is close to μ_1. If so, we can put $x_1^* = \mu_1 + \epsilon$, and use the approximation

$$P(x_1^*) \approx P(\mu_1), \qquad Q(x_1^*) \approx Q(\mu_1) + Q'(\mu_1)\epsilon.$$

Then

$$x_1 \gtrsim \mu_1 + \min_{\epsilon>0}\left(\epsilon + \sqrt{\frac{P(\mu_1)}{Q'(\mu_1)\epsilon}}\right) = \mu_1 + 3\left(\frac{P(\mu_1)}{4Q'(\mu_1)}\right)^{1/3}.$$

Similarly, for the largest zero one would expect (notice that $Q'(\mu_2) < 0$)

$$x_k \lesssim \mu_2 + 3\left(\frac{P(\mu_2)}{4Q'(\mu_2)}\right)^{1/3}.$$

We will show in the following sections that bounds of this type hold indeed for the classical discrete polynomials, although sometimes we are able to obtain only a weaker constant before the second order term.

Concerning (7) and (8) we just note that the minimization problem is equivalent to finding $\min A^2/B$.

The intersection conditions imposed in Theorem 2 do not yield any severe restrictions. Of course, if t_0 is continuous, it intersects all the interior branches $\mathcal{B}_1, \ldots, \mathcal{B}_{k-1}$, thus, at worst, giving bounds for x_2 and x_{k-1}. As we will see, in the classical case, t_0 intersects $\mathcal{B}_0$ and $\mathcal{B}_k$ whenever the polynomial has no zeros in $(x_0, x_0+1]$ and $[x_{k+1}-1, x_{k+1})$, respectively. In fact, the following two simple lemmas will suffice for checking the intersection conditions.

Lemma 3. *If $\mathcal{M}(p) > 1$ then $t(x)$ is an increasing function of x.*

Proof. It is enough to show that in the partial fraction decomposition $\frac{p(x-1)}{p(x)} = 1 + \sum_{i=1}^{k} \frac{\lambda_i}{x-x_i}$, all λ_i are negative. We have

$$\lambda_i = \lim_{x\to x_i} \frac{(x-x_i)p(x-1)}{p(x)}.$$

Let, say, $x_i < x < x_i + 1 < x_{i+1}$. Since $\mathcal{M}(p) > 1$ the zeros of $p(x)$ and $p(x-1)$ are interlacing. Hence, $\operatorname{sgn} p(x-1) = -\operatorname{sgn} p(x)$ and the result follows. □

Lemma 4. *Suppose that $p(x)$ is orthogonal on $L = \{0, 1, 2, \ldots, x_{k+1}\}$, and satisfies the difference equation*

$$C(x)p(x+1) = 2A_1(x)p(x) - B_1(x)p(x-1), \tag{10}$$

where $A_1(x)$, $B_1(x)$, $C(x)$ are strictly positive and continuous on $(0, x_{k+1})$.

(i) *If $A_1(0) > 0$, $B_1(0) = 0$, $C(0) > 0$, then y_0 exists. Moreover, $p(x)$ has no zeros in $[0, 1]$.*

(ii) *If* $A_1(x_{k+1}) > 0$, $B_1(x_{k+1}) > 0$, $C(x_{k+1}) = 0$, *then* y_k *exists. Moreover, if* $x_{k+1} = n$ *is finite, then* $p(x)$ *has no zeros in* $[n-1, n]$.

Proof. Since $p(x)$ has no zeros for $x \le 0$ we have $0 < t(0) < \infty$, by $t(0) = \frac{p(-1)}{p(0)}$. Now the existence of y_0 follows by

$$t(x_1^-) = \infty, \qquad t_0(0^+) = \infty, \qquad t_0(x_1) < \infty.$$

Furthermore, as $\frac{A_1(0)}{C(0)} > 0$ then $\operatorname{sgn} p(1) = \operatorname{sgn} p(0)$. Hence, $p(x)$ has an even number of zeros in $(0, 1)$. Since the zeros must be separated by a point of the support of the measure we conclude that there are no zeros of $p(x)$ on $[0, 1]$.

To prove (ii) we find by (10), $p(n-1) = \frac{2A_1(n)}{B_1(n)} p(n)$, that is

$$t(n) = \frac{2A_1(n)}{B_1(n)} = 2t_0(n) > 0.$$

The existence of y_k is a consequence of

$$t(x_k^+) = -\infty, \qquad t_0(x_k) > -\infty, \qquad t(x_{k+1}) > t_0(x_{k+1}).$$

The absence of zeros on $[n-1, n]$ follows as above. □

Remark 2. It is worth noticing that in the opposite direction one has:

if $A_1(0) \le 0$, $B_1(0) = 0$, $C(0) > 0$, then $p(x)$ has a zero on $(0, 1]$.

If $x_{k+1} = n$ is finite and $A_1(n) \le 0$, $B_1(n) > 0$, $C(n) = 0$, then $p(x)$ has a zero on $[n-1, n]$.

For Charlier, Krawtchouk and Meixner polynomials the function A^2/B, bounding the zero spacing, is unimodal. The situation in the Hahn case is more subtle. In particular, depending on the parameters A^2/B may be either unimodal with the minimum attained in the bulk of $[\mu_1, \mu_2]$ or has two minima around μ_1 and μ_2. The equation for the discriminant surface separating these cases turns out to be quite complicated. Here we will give bounds for the mesh in the symmetric case only.

In what follows we prefer dealing with Eq. (1) rather than (10). When it is necessary, the choice of C will be the obvious simplest one.

3. Charlier polynomials

The Charlier polynomials $C_k(x) = C_k^a(x)$ are orthogonal on the integer points of $[0, \infty)$ for $a > 0$. They satisfy difference equation (1) with

$$A = \frac{x + a - k}{2a}, \qquad B = \frac{x}{a}.$$

Thus,

$$t_0 = \frac{x + a - k}{2x}, \qquad \Delta = -\frac{x^2 - 2x(a+k) + (a-k)^2}{4a^2}.$$

The solutions of $\Delta(x_0) = 0$, giving the first order bounds are

$$\mu_{1,2} = \left(\sqrt{k} \pm \sqrt{a}\right)^2 \subset \bar{\mathcal{L}} = [0, \infty).$$

Let us also notice that $2A(0) = 1 - \frac{k}{a}$. Thus, for $k \ge a$ the polynomial has a zero on $[0, 1)$.

The following theorem improves slightly the inequalities given in [6] and also removes some unnecessary restrictions.

Theorem 5.

$$x_1 > (\sqrt{a} - \sqrt{k}\,)^2 + \frac{3 \cdot 2^{-2/3} a^{1/6} (\sqrt{a} - \sqrt{k}\,)^{2/3}}{k^{1/6}}, \tag{11}$$

provided $a > k$.

$$x_k < (\sqrt{a} + \sqrt{k}\,)^2 - \frac{3 \cdot 2^{-2/3} a^{1/6} (\sqrt{a} + \sqrt{k}\,)^{2/3}}{k^{1/6}}, \tag{12}$$

$$\mathcal{M}(C_k^a) \geq 2\sqrt{\frac{a + \sqrt{a(a-k)}}{k}}, \tag{13}$$

provided $a > k$.

Proof. The intersection condition in this case is just $a > k$. This readily follows from Lemma 4. To prove (11) we set the extreme value of x in (5) to $\mu_1 + \epsilon, \quad 0 < \epsilon < \mu_2 - \mu_1$. This gives

$$\begin{aligned} x_1 &> \mu_1 + \min_\epsilon \left(\epsilon + 2\sqrt{\frac{a(\mu_1 + \epsilon)}{\epsilon(\mu_2 - \mu_1 - \epsilon)}} \right) \\ &> \mu_1 + \min_\epsilon \left(\epsilon + 2\sqrt{\frac{a\mu_1}{\epsilon(\mu_2 - \mu_1)}} \right) = \mu_1 + 3\left(\frac{a\mu_1}{\mu_2 - \mu_1} \right)^{1/3} \\ &= (\sqrt{a} - \sqrt{k}\,)^2 + \frac{3 \cdot 2^{-2/3} a^{1/6} (\sqrt{a} - \sqrt{k}\,)^{2/3}}{k^{1/6}}. \end{aligned}$$

The proof of (12) is similar.

To prove (13) we use the substitution $x = (\sqrt{a} - \epsilon\sqrt{k}\,)^2, \quad |\epsilon| < 1$. The condition $a > k$ implies that $A(x) > 0$. Hence $\mathcal{M}(C_k^a) > 2$, by Theorem 1. Now, we get from (8),

$$\mathcal{M}(C_k^a) \geq 2 \min_x \frac{\sqrt{2}\, a^{1/4} k^{1/4}}{\sqrt{2\sqrt{ax} - x - a + k}} = 2\sqrt{2}\left(\frac{a}{k}\right)^{1/4} \min_{|\epsilon|<1} \sqrt{\frac{\sqrt{\frac{a}{k}} - \epsilon}{1 - \epsilon^2}}.$$

We find the derivative

$$\frac{\partial}{\partial \epsilon} \frac{\sqrt{\frac{a}{k}} - \epsilon}{1 - \epsilon^2} = \frac{2\sqrt{\frac{a}{k}}\,\epsilon - \epsilon^2 - 1}{(1 - \epsilon^2)^2},$$

with the zeros $\epsilon_1 = \frac{\sqrt{a} - \sqrt{a-k}}{\sqrt{k}}$ and $\epsilon_2 = \frac{\sqrt{a} + \sqrt{a-k}}{\sqrt{k}} > 1$. This implies that the minimum is attained for $\epsilon = \epsilon_1$, yielding (13). $\square$

Remark 3. Probably, a more convenient choice of the parameters in the Charlier case is k and $\beta = \sqrt{a/k}$. Using this one can rewrite (11)–(13) as follows:

$$x_1 > k(\beta - 1)^2 \left(1 + \frac{3\beta^{1/3}}{2^{2/3}(\beta - 1)^{4/3} k^{2/3}} \right),$$

$$x_k < k(\beta + 1)^2 \left(1 - \frac{3\beta^{1/3}}{2^{2/3}(\beta + 1)^{4/3} k^{2/3}} \right),$$

$$\mathcal{M}(C_k^a) \geq 2\sqrt{\beta^2 + \beta\sqrt{\beta^2 - 1}},$$

and $a > k$ means $\beta > 1$.

Remark 4. There is a simple way to check the validity of Conjectures 1 and 2 for a certain range of the parameters. In the Charlier case this can be done using the limiting relation between Charlier and Hermite polynomials (see e.g. [4]),

$$\lim_{a\to\infty} (2a)^{k/2} C_k^a(\sqrt{2a}\,x) = (-1)^k H_k(x),$$

and the well-known bounds on the zeros of H_k and their spacing (see e.g. [15]). This shows that (11)–(13) are of the right order for large a. Similar arguments are applicable for other classical discrete orthogonal polynomials.

Remark 5. The orthogonality provides a sufficient but clearly not a necessarily condition for a polynomial to have only real distinct zeros. For example $C_2^a(x) = (x-a)^2 - x$, and has two zeros $x_1, x_2 \in (0,\infty)$ for $a > -1/4,\ a \neq 0$. It is worth noticing that our method still works in that case.

4. Krawtchouk polynomials

The Krawtchouk polynomials $K_k(x) = K_k^n(x;q), k = 0, 1, \ldots, n$, are orthogonal on the integer points of $[0,n]$ for $0 < q < 1$. They satisfy difference equation (1) with

$$A = \frac{q(n-x) + (1-q)x - k}{2q(n-x)}, \qquad B = \frac{x(1-q)}{q(n-x)}.$$

This gives

$$t_0 = \frac{q(n-x) + (1-q)x - k}{2x(1-q)},$$

$$\Delta = -\frac{x^2 - 2x(nq - 2kq + k) + (nq-k)^2}{4q^2(n-x)^2} := \frac{P(x)}{Q(x)}.$$

The equation $\Delta(x) = 0$ has the solutions $\mu_1 < \mu_2$,

$$\mu_{1,2} = \left(\sqrt{q(n-k)} \pm \sqrt{(1-q)k}\right)^2.$$

Clearly, $\mu_{1,2} \in \bar{\mathcal{L}} = [0,n]$, since

$$P(0) = -(nq-k)^2 \leq 0, \qquad P(n) = -(n(1-q)-k)^2 \leq 0.$$

We also have with $C = q(n-x)$ in (10),

$$2A_1(0) = qn - k, \qquad 2A_1(n) = (1-q)n - k,$$

meaning by Lemma 4 that for $k \geq qn$ the polynomial has a zero on $(0,1]$, and for $k > (1-q)n$ on $[n-1,n)$.

Theorem 6.

$$x_1 > \mu_1 + \frac{3q^{1/6}(1-q)^{1/6}\mu_1^{1/3}(n-\mu_1)^{1/3}}{2^{2/3}k^{1/6}(n-k)^{1/6}}, \tag{14}$$

provided $k < qn$;

$$x_k < \mu_2 - \frac{3q^{1/6}(1-q)^{1/6}\mu_2^{1/3}(n-\mu_2)^{1/3}}{2^{2/3}k^{1/6}(n-k)^{1/6}}, \tag{15}$$

provided $k < (1-q)n$;

$$\mathcal{M}(K_k) \geq 2\sqrt{u^2 + u\sqrt{u^2-1}}, \quad u = n\sqrt{\frac{q(1-q)}{k(n-k)}}, \tag{16}$$

provided $k < n\min(q, 1-q)$.

Proof. Lemma 4 yields that y_0 and y_k exist for $k < qn$ and $k < (1-q)n$, respectively.

To prove (14) we set the extremal value of x in (5) to $\mu_1 + \epsilon$, $0 < \epsilon < \mu_2 - \mu_1$, yielding

$$\begin{aligned}
x_1 &> \mu_1 + \min_{0<\epsilon<\mu_2-\mu_1}\left(\epsilon + 2\sqrt{\frac{q(1-q)(\mu_1+\epsilon)(n-\mu_1-\epsilon)}{\epsilon(\mu_2-\mu_1-\epsilon)}}\right) \\
&> \mu_1 + \min_{0<\epsilon<\mu_2-\mu_1}\left(\epsilon + 2\sqrt{\frac{q(1-q)\mu_1(n-\mu_1-\epsilon)}{\epsilon(\mu_2-\mu_1-\epsilon)}}\right) \\
&> \mu_1 + \min_{0<\epsilon<\mu_2-\mu_1}\left(\epsilon + 2\sqrt{\frac{q(1-q)\mu_1(n-\mu_1)}{\epsilon(\mu_2-\mu_1)}}\right) \\
&= \mu_1 + 3\left(\frac{q(1-q)\mu_1(n-\mu_1)}{(\mu_2-\mu_1)}\right)^{1/3}.
\end{aligned}$$

One can readily check that the last expression is equal to (14). The proof of (15) is similar.

Let us prove (16). Notice that by Theorem 1 the condition $k < n\min(q, 1-q)$ implies $\mathcal{M}\left(K_k^n(x; q)\right) > 2$. We have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sqrt{B(x)}}{\sqrt{B(x)} - A(x)} = \frac{2B(x)A'(x) - B'(x)A(x)}{2\sqrt{B(x)}\,(\sqrt{B(x)} - A(x))^2}, \\
&2B(x)A'(x) - B'(x)A(x) = \frac{(1-q)((n-2k)x - n(nq-k))}{2q^2(n-x)^3}.
\end{aligned}$$

Hence, the minimum of the function is attained at $x = \frac{n(nq-k)}{(1-q)(n-2k)}$. We find by (8),

$$\mathcal{M}(K_k) \geq 2q^{1/4}(1-q)^{1/4}\sqrt{\frac{n\left(n\sqrt{q(1-q)} + \sqrt{(nq-k)(n(1-q)-k)}\right)}{k(n-k)}}.$$

The last expression is equivalent to (16). This completes the proof. □

In applications the most important case is the binary Krawtchouk polynomials corresponding to $q = \frac{1}{2}$. They are symmetric with respect to $\frac{n}{2}$, so $x_k = \frac{n}{2} - x_1$, and Theorem 6 gives:

Corollary 1. *Suppose that* $k < \frac{n}{2}$. *Then*

$$x_1 > \frac{n}{2} - \sqrt{k(n-k)}\left(1 - \frac{3}{2}\left(\frac{n-2k}{2k(n-k)}\right)^{2/3}\right), \tag{17}$$

$$\mathcal{M}\left(K_k^n\left(x;\frac{1}{2}\right)\right) \geq \sqrt{\frac{2n}{k}}. \tag{18}$$

Inequality (17) improves slightly the bound given in [5]. In the opposite direction. Levenshtein proved [10],

$$x_1 < \frac{n}{2} - \sqrt{k(n-k)} + k^{1/6}\sqrt{n-k}, \quad k \leq [n/2].$$

This seems to be the only known result of such a type.

Remark 6. The formulas can be simplified if we choose as the parameters ϕ and θ; $0 \leq \phi \leq \frac{\pi}{2}$, $0 < \theta < \frac{\pi}{2}$; defined by

$$k = n\sin^2\phi, \qquad q = \sin^2\theta.$$

Bounds (14) and (16) become

$$x_1 > n\sin^2(\theta-\phi)\left(1 + 3\left(\frac{\cos^2(\theta-\phi)\sin 2\theta}{4n^2\sin^4(\theta-\phi)\sin 2\phi}\right)^{1/3}\right), \quad \phi < \theta;$$

$$x_k < n\sin^2(\theta+\phi)\left(1 - 3\left(\frac{\cos^2(\theta+\phi)\sin 2\theta}{4n^2\sin^4(\theta+\phi)\sin 2\phi}\right)^{1/3}\right), \quad \phi < \frac{\pi}{2} - \theta.$$

In connection with (16) we just notice that $u = \frac{\sin 2\theta}{\sin 2\phi}$.

5. Meixner polynomials

The Meixner polynomials $M_k(x) = M_k(x;\beta,c)$ are orthogonal on the integer points of $[0,\infty)$ for $\beta > 0$ and $0 < c < 1$. They satisfy difference equation (1) with

$$A = \frac{x(1+c)+\beta c - k(1-c)}{2c(x+\beta)}, \qquad B = \frac{x}{c(x+\beta)}.$$

This gives

$$t_0 = \frac{x(1+c)+\beta c - k(1-c)}{2x},$$

$$\Delta = -\frac{(1-c)^2x^2 - 2(1-c)\left(k(1+c)+\beta c\right)x + ((1-c)k-\beta c)^2}{4c^2(x+\beta)^2} := \frac{P(x)}{Q(x)}.$$

The equation $\Delta(x) = 0$ has the solutions $\mu_1 < \mu_2$,

$$\mu_{1,2} = \frac{\left(\sqrt{k} \pm \sqrt{c(k+\beta)}\right)^2}{1-c}.$$

The condition $\mu_{1,2} \in [x_0, x_{k+1}] = [0,\infty)$, follows by $\mu_1 + \mu_2 > 0$ and $P(0) \leq 0$.

We have

$$2A(0) = 1 - \frac{(1-c)k}{\beta c},$$

hence for $k \geq \frac{\beta c}{1-c}$ the polynomial has a zero on $(0, 1]$.

Theorem 7.

$$x_1 > \mu_1 + \frac{3c^{1/6}\mu_1^{1/3}(\mu_1+\beta)^{1/3}}{2^{2/3}(1-c)^{1/3}k^{1/6}(k+\beta)^{1/6}}, \tag{19}$$

provided $k < \frac{\beta c}{1-c}$;

$$x_k < \mu_2 - \begin{cases} \dfrac{3c^{1/6}\mu_2^{1/3}(\mu_2+\beta)^{1/3}}{2^{2/3}(1-c)^{1/3}k^{1/6}(k+\beta)^{1/6}}, & \mu_2 \le \mu_1 + \sqrt{\mu_1(\mu_1+\beta)}, \\ \dfrac{3c^{1/3}(\sqrt{\mu_1} + \sqrt{\mu_1+\beta})^{2/3}}{(1-c)^{2/3}}, & \mu_2 > \mu_1 + \sqrt{\mu_1(\mu_1+\beta)}, \end{cases} \tag{20}$$

provided $k > 1$;

$$\mathcal{M}(M_k) \ge \frac{2\sqrt{\beta^2 c + \beta\sqrt{c(\beta c - (1-c)k)(\beta + (1-c)k)}}}{(1-c)\sqrt{k(k+\beta)}}. \tag{21}$$

provided $k < \frac{\beta c}{1-c}$.

Proof. By Lemma 4, y_k always exists, and the only condition required for the existence of y_0 is $k < \frac{\beta c}{1-c}$. The proofs of (19) and (21) are similar to the previous cases and will be omitted. Let us demonstrate (20).

Set the extremal value of x in (6) to $\mu_2 - \epsilon,\ 0 < \epsilon < \mu_2 - \mu_1$. This gives

$$x_k < \max_x \left(x - \sqrt{\frac{B(x)}{\Delta(x)}} \right) = \mu_2 - \min_\epsilon \left(\epsilon + \frac{2\sqrt{c}}{1-c} \sqrt{\frac{(\mu_2-\epsilon)(\mu_2+\beta-\epsilon)}{\epsilon(\mu_2-\mu_1-\epsilon)}} \right).$$

To estimate the minimum we apply the inequality

$$x + y \ge 3 \cdot 2^{-2/3} x^{1/3} y^{2/3}, \quad x, y \ge 0,$$

holding by $4(x+y)^3 - 27xy^2 = (2x-y)^2(x+4y) \ge 0$.

We obtain

$$\min_\epsilon \left(\epsilon + \frac{2\sqrt{c}}{1-c} \sqrt{\frac{(\mu_2-\epsilon)(\mu_2+\beta-\epsilon)}{\epsilon(\mu_2-\mu_1-\epsilon)}} \right) \ge 3 \left(\frac{c\,(\mu_2-\epsilon)(\mu_2+\beta-\epsilon)}{(1-c)^2\,(\mu_2-\mu_1-\epsilon)} \right)^{1/3},$$

and

$$(\mu_2-\mu_1-\epsilon)^2 \frac{\mathrm{d}}{\mathrm{d}x} \frac{(\mu_2-\epsilon)(\mu_2+\beta-\epsilon)}{\mu_2-\mu_1-\epsilon} = -\epsilon^2 + 2(\mu_2-\mu_1)\epsilon - \mu_2^2 + 2\mu_1\mu_2 + \beta\mu_1.$$

The zeros of the derivative are

$$\epsilon_1 = \mu_2 - \mu_1 - \sqrt{\mu_1(\mu_1+\beta)}, \qquad \epsilon_2 = \mu_2 - \mu_1 + \sqrt{\mu_1(\mu_1+\beta)} > \mu_2 - \mu_1.$$

Thus, the sought minimum is attained at $\epsilon = 0$, if $\epsilon_1 < 0$ and at $\epsilon = \epsilon_1$, otherwise. In the first case calculations yield

$$x_k < \mu_2 - \frac{3c^{1/6}\mu_2^{1/3}(\mu_2+\beta)^{1/3}}{2^{2/3}(1-c)^{1/3}k^{1/6}(k+\beta)^{1/6}},$$

and in the second

$$x_k < \mu_2 - \frac{3c^{1/3}(\sqrt{\mu_1} + \sqrt{\mu_1+\beta})^{2/3}}{(1-c)^{2/3}}. \quad \square$$

We will show that two possible second order terms in (20) are pretty close. But first we have to clarify the condition

$$\mu_2 > \mu_1 + \sqrt{\mu_1(\mu_1 + \beta)}. \tag{22}$$

For this purpose, we introduce the parameters γ and z defined by

$$c = \gamma^2, \quad 0 < \gamma < 1; \qquad \beta = k(z^2 - 1), \quad z > 1. \tag{23}$$

Then the requirement $k < \frac{\beta c}{1-c}$ becomes just $\gamma z > 1$. Using this one obtains that (22) is equivalent to

$$g(\gamma, z) = \gamma^2 z - (z^2 - 4z + 1)\gamma + z > 0.$$

This yields the following system of constrains

$$\begin{cases} \gamma z > 1, \\ z > 3 + 2\sqrt{2}, \\ \gamma < \dfrac{1 - 4z + z^2 - (z-1)\sqrt{z^2 - 6z + 1}}{2z}. \end{cases} \tag{24}$$

Now we can present another (slightly weaker) inequality on the largest zero x_k of the Meixner polynomial which is simpler and preserves more symmetry with (19).

Theorem 8. *For $k > 1$,*

$$x_k < \mu_2 - \frac{6^{1/3} c^{1/6} \mu_2^{1/3} (\mu_2 + \beta)^{1/3}}{(1-c)^{1/3} k^{1/6} (k + \beta)^{1/6}}. \tag{25}$$

Proof. Consider

$$G_1 = \frac{3c^{1/6} \mu_2^{1/3} (\mu_2 + \beta)^{1/3}}{2^{2/3} (1-c)^{1/3} k^{1/6} (k+\beta)^{1/6}}, \qquad G_2 = \frac{3c^{1/3} \left(\sqrt{\mu_1} + \sqrt{\mu_1 + \beta}\right)^{2/3}}{(1-c)^{2/3}},$$

the second order terms in (20). We will show that for $\mu_2 > \mu_1 + \sqrt{\mu_1(\mu_1 + \beta)}$ the ratio G_1/G_2 satisfies

$$1 < \frac{G_1}{G_2} < \frac{9^{1/3}}{2}, \tag{26}$$

implying (25).

The inequality $G_1/G_2 > 1$ is obvious, as G_2 was obtained as an absolute minimum of the corresponding function.

To prove the second one we use the change of variables defined by (23). We find

$$\left(\frac{G_1}{G_2}\right)^3 = \frac{(z+\gamma)^2 (1+\gamma z)^2}{4\gamma z (1+\gamma)^2 (z-1)^2},$$

$$\frac{\partial}{\partial \gamma} \left(\frac{G_1}{G_2}\right)^3 = -\frac{(1-\gamma)(\gamma+z)(1+\gamma z)}{4\gamma^2 z (1+\gamma)^3 (z-1)^2}\, g(\gamma, z) < 0,$$

$$\frac{\partial}{\partial z} \left(\frac{G_1}{G_2}\right)^3 = -\frac{(1+z)(\gamma+z)(1+\gamma z)}{4\gamma z^2 (1+\gamma)^2 (z-1)^3}\, g(\gamma, z) < 0.$$

Hence, by (24) the maximum is attained for $\gamma = 1/z$, $z = 3 + 2\sqrt{2}$ giving (26). $\square$

Remark 7. In terms of (23) we can restate (19), (25) and (21) in a more transparent form as follows:

$$x_1 > \frac{(\gamma z-1)^2 k}{1-\gamma^2}\left(1+\frac{3\gamma^{1/3}(z-\gamma)^{2/3}}{2^{2/3}z^{1/3}(\gamma z-1)^{4/3}k^{2/3}}\right),$$
$$x_k < \frac{(1+\gamma z)^2 k}{1-\gamma^2}\left(1-\frac{6^{1/3}\gamma^{1/3}(\gamma+z)^{2/3}}{z^{1/3}(1+\gamma z)^{4/3}k^{2/3}}\right),$$
$$\mathcal{M}(M_k) > \frac{2\sqrt{\gamma(z^2-1)}}{(1-\gamma^2)z}\sqrt{\gamma(z^2-1)+\sqrt{(z^2-\gamma^2)(\gamma^2z^2-1)}},$$

provided $\gamma z > 1$.

6. Hahn polynomials $\alpha, \beta > -1$

The Hahn polynomials $Q_k(x) = Q_k^n(x;\alpha,\beta)$, $k = 0, 1, \dots, n$, are orthogonal on the integer points of $[0, n]$ for $\alpha, \beta > -1$ or $\alpha, \beta < -n$. In the last case they are occasionally called the Hahn–Eberlein polynomials. The Hahn polynomials satisfy difference equation (1) with

$$A = \frac{(n-x)(x+\alpha+1)+x(n+\beta+1-x)-k(k+\alpha+\beta+1)}{2(n-x)(x+\alpha+1)},$$
$$B = \frac{x(n+\beta+1-x)}{(n-x)(x+\alpha+1)}.$$

Notice that $B(x)$ may by negative if $< -n-1 < \max(\alpha,\beta) < -n$. Since (see e.g. [14]),

$$Q_k^n(x;\alpha,\beta) = (-1)^k\frac{(\beta+1)_k}{(\alpha+1)_k}Q_k^n(n-x;\beta,\alpha),$$

without loss of generality we will restrict ourselves to two cases $\alpha \ge \beta > -1$, which we consider in this section, and $\alpha < \beta < -n-1$, which will be discussed in the next one. Bounds for the mesh in the symmetric case $\alpha = \beta$ will be given in the last section. Concerning the general case, let us just notice that since $A(0) \ge A(n)$ by the assumption $\beta \le \alpha$, and since the function $A_1(x)$ has no local minima, the conditions $A_1(0) > 0$ and $A_1(n) > 0$ will suffice to ensure that the mesh of a Hahn polynomial is greater than two.

First, we need a preparatory lemma which will be easier to deduce from the three term recurrence relation than from the difference equation.

Lemma 9. *If $\alpha \ge \beta > -1$ or $\alpha \le \beta \le -n-1$, then*

$$x_1 < \frac{(\alpha+k)(n-k+1)}{2k+\alpha+\beta} < \frac{(\alpha+k)n+(b+k)(k-1)}{2k+\alpha+\beta} < x_k. \tag{27}$$

Proof. Suppose that a family of monic orthogonal polynomials satisfies the three term recurrence

$$p_{i+1}(x) = (x-a_i)p_i(x) - b_i p_{i-1}(x).$$

By comparing the coefficients at x^i one readily obtains for the zeros $x_1 < x_2 < \cdots < x_k$ of p_k,

$$\sum_{i=1}^{k} x_i = \sum_{i=0}^{k-1} a_i.$$

The coefficients of the three term recurrence in the Hahn case are

$$a_i = \frac{(i+\alpha+\beta+1)(i+\alpha+1)(n-i)}{(2i+\alpha+\beta+1)(2i+\alpha+\beta+2)} + \frac{i(i+\beta)(n+i+\alpha+\beta+1)}{(2i+\alpha+\beta)(2i+\alpha+\beta+1)}$$
$$= \frac{2n-\alpha+\beta}{4} + \frac{(\alpha^2-\beta^2)(2n+\alpha+\beta+2)}{8}\left(\frac{1}{2i+\alpha+\beta} - \frac{1}{2i+\alpha+\beta+2}\right).$$

Hence

$$\bar{x} = \frac{1}{k}\sum_{i=1}^{k} x_i = \frac{1}{k}\sum_{i=0}^{k-1} a_i = \frac{n}{2} + \frac{(\alpha-\beta)(n-k+1)}{2(2k+\alpha+\beta)} \geq \frac{n}{2}.$$

Since the mesh is greater than one, then $x_1 < \bar{x} - \frac{k-1}{2}, \quad x_k > \bar{x} + \frac{k-1}{2}$. □

We have

$$t_0 = \frac{-2x^2 + (2n-\alpha+\beta)x + S_0}{2x(n+\beta+1-x)},$$
$$\Delta = -\frac{S_2x^2 - 2S_1x + S_0^2}{4(n-x)^2(x+\alpha+1)^2},$$

where

$$S_0 = (1+\alpha)n - k(k+\alpha+\beta+1), \tag{28}$$
$$S_1 = ((2k+\alpha+1)n - (\alpha-\beta)k)\,(k+\alpha+\beta+1) - (1+\alpha)(k-1)n, \tag{29}$$
$$S_2 = (2k+\alpha+\beta+2)^2 - 4k > 0. \tag{30}$$

To simplify otherwise messy formulas it will be convenient to introduce the following variables:

$$r = k(k+\alpha+\beta+1), \tag{31}$$
$$q = n+\alpha+\beta+2, \tag{32}$$
$$V_1 = \sqrt{r+(1+\alpha)(1+\beta)}, \tag{33}$$
$$V_2 = \sqrt{r(qn-r)}, \tag{34}$$
$$U_1 = n(q-n) - 2r, \tag{35}$$
$$U_2 = q(q-n) + 2r. \tag{36}$$

The equation $\Delta(x) = 0$ has the solutions $\mu_1 < \mu_2$,

$$\mu_{1,2} = \frac{S_1 \pm 2V_1V_2}{S_2}.$$

To show that $\mu_{1,2} \in \bar{\mathcal{L}} = [0, n]$, we set

$$P(x) = 4(n-x)^2(x+\alpha+1)^2\Delta = -S_2x^2 + 2S_1x - S_0^2,$$

and find

$$P(0) = -S_0^2 \leq 0, \qquad P(n) = -\left(n(1+\beta) - r\right)^2 \leq 0,$$
$$\mu_1\mu_2 = \frac{S_0^2}{S_2} \geq 0.$$

We also have with $C = (n-x)(x+\alpha+1)$ in (10),

$$2A_1(0) = (1+\alpha)n - r, \qquad 2A_1(n) = (1+\beta)n - r,$$

providing the explicit form for intersection conditions $A_1(0) > 0$, $A_1(n) > 0$.

Theorem 10. *Suppose $\alpha \ge \beta > -1$. Then*

$$x_1 > \frac{S_1 - 2V_1V_2}{S_2} + \frac{3}{2S_2}\left(\frac{((\alpha-\beta)V_2 + U_1V_1)^2\,((\alpha-\beta)V_2 + U_2V_1)^2}{5V_1V_2S_2}\right)^{1/3}, \tag{37}$$

provided $k(k+\alpha+\beta+1) < n(1+\alpha)$ and $k \ge 5$.

$$x_k < \frac{S_1 + 2V_1V_2}{S_2} - \frac{3}{2S_2}\left(\frac{((\alpha-\beta)V_2 - U_1V_1)^2\,((\alpha-\beta)V_2 - U_2V_1)^2}{V_1V_2S_2}\right)^{1/3}, \tag{38}$$

provided $k(k+\alpha+\beta+1) < n(1+\beta)$.

Proof. The existence of y_0 and y_k under the imposed restrictions follows from Lemma 4.

We start with proving (38). From (27) it follows that $x_k > \frac{n}{2}$, and we can apply (6) with $\nu_2 = \frac{n}{2}$. Setting $x = \mu_2 - \epsilon$, $\;0 < \epsilon < \mu_2 - \frac{n}{2}$, we have to estimate

$$\begin{aligned}\frac{B(x)}{\Delta(x)} &= \frac{4x(n-x)(x+\alpha+1)(n+\beta+1-x)}{-2S_0^2 + 4S_1x - 2S_2x^2}\\ &= \frac{2x(n-x)(x+\alpha+1)(n+\beta+1-x)}{(x-\mu_1)(\mu_2-x)S_2} := \frac{2T(x)}{\epsilon(x-\mu_1)S_2}.\end{aligned} \tag{39}$$

The functions $x(n-x)$ and $\frac{(x+\alpha+1)(n+\beta+1-x)}{x-\mu_1}$ are both decreasing in x for $\frac{n}{2} < x < \mu_2$. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{(x+\alpha+1)(n+\beta+1-x)}{x-\mu_1} = -\frac{(\mu_1-x)^2 + (\mu_1+\alpha+1)(n+\beta+1-\mu_1)}{(x-\mu_1)^2} < 0.$$

Hence,

$$\begin{aligned}\frac{B(x)}{\Delta(x)} &\ge \frac{2\mu_2(n-\mu_2)(x+\alpha+1)(n+\beta+1-x)}{\epsilon(x-\mu_1)S_2}\\ &\ge \frac{2\mu_2(n-\mu_2)(\mu_2+\alpha+1)(n+\beta+1-\mu_2)}{\epsilon(\mu_2-\mu_1)S_2} = \frac{2T(\mu_2)}{\epsilon(\mu_2-\mu_1)S_2} = \frac{T(\mu_2)}{2\epsilon V_1V_2}.\end{aligned} \tag{40}$$

This gives

$$x_k < \mu_2 - \min_{0<\epsilon<\mu_2-\frac{n}{2}}\left(\epsilon + \sqrt{\frac{T(\mu_2)}{2\epsilon V_1V_2}}\right) = \mu_1 - 3\left(\frac{T(\mu_2)}{4V_1V_2}\right)^{1/3}. \tag{41}$$

The following identity can be checked directly:

$$\begin{aligned}T(\mu_2) &= \mu_2(n-\mu_2)(\mu_2+\alpha+1)(n+\beta+1-\mu_2)\\ &= ((\alpha-\beta)V_2 - U_1V_1)^2\,((\alpha-\beta)V_2 - U_2V_1)^2\,S_2^{-4}.\end{aligned} \tag{42}$$

Together with (41) this yields (38).

To prove (37) we observe first that

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{(x+\alpha+1)(n-x)}{\mu_2-x}=\frac{(\mu_2-x)^2+(n-\mu_2)(\mu_2+\alpha+1)}{(\mu_2-x)^2}>0.$$

Set as above $x=\mu_1+\epsilon$, where by (27) we may restrict ourselves to $0<\epsilon<\hat{x}-\mu_1$, $\quad \hat{x}=\frac{(\alpha+k)(n-k+1)}{2k+\alpha+\beta}$. We have similarly to (39),

$$\frac{B(x)}{\Delta(x)}>x(n+\beta+1-x)\,\frac{4(n-\mu_1)(\mu_1+\alpha+1)}{\epsilon(\mu_2-\mu_1)S_2^2}.$$

The minimum of this function is attained either for $x=\mu_1$ or for $x=\hat{x}$. Put $\alpha_1=\alpha+1\geq 0$, $\quad \beta_1=\beta+1\geq 0$. Let us show that the ratio

$$\tau=\frac{\mu_1(n+\beta+1-\mu_1)}{\hat{x}(n+\beta+1-\hat{x})}<\frac{n+\beta_1-\mu_1}{n+\beta_1-\hat{x}}<5. \tag{43}$$

First, we need the estimate

$$\begin{aligned} n+\beta_1-\mu_1 &= \frac{(n+\beta_1)S_2-S_1+2V_1V_2}{S_2}<\frac{2(n+\beta_1)S_2-2S_1}{S_2}\\ &< \frac{4(k+\alpha_1)(k+\beta_1)(n+\alpha_1+\beta_1)}{S_2}. \end{aligned} \tag{44}$$

The first inequality in (44) follows by

$$((n+\beta_1)S_2-S_1)^2-4V_1^2V_2^2=(r+q\beta_1)^2\,S_2>0,$$

that is, by

$$(n+\beta_1)S_2-S_1>2V_1V_2>0.$$

The second one follows by the identity

$$\begin{aligned} (n+\beta_1)S_2-S_1 &= 2(k+\alpha_1)(k+\beta_1)(n+\alpha_1+\beta_1)-k\left(2n+\alpha_1(k+1)-\beta_1(k-3)\right)\\ &\quad -(\alpha_1-\beta_1)\left((\alpha_1+\beta_1)(k+\beta_1)+n\beta_1\right). \end{aligned}$$

Now, we find

$$\begin{aligned} \tau &< \frac{4(k+\alpha_1)(k+\beta_1)(n+\alpha_1+\beta_1)}{(n+\beta_1-\hat{x})S_2}\\ &= \frac{4(k+\alpha_1)(k+\beta_1)(2k+\alpha_1+\beta_1-2)(n+\alpha_1+\beta_1)}{\left((2k+\alpha_1+\beta_1)^2-4k\right)(k+\beta_1-1)(n+k+\alpha_1+\beta_1-1)}\\ &< \frac{4(k+\alpha_1)(k+\beta_1)(2k+\alpha_1+\beta_1-2)}{\left((2k+\alpha_1+\beta_1)^2-4k\right)(k+\beta_1-1)}<5, \end{aligned}$$

provided $k\geq 5$.

Thus, we obtain

$$\frac{B(x)}{\Delta(x)}>\frac{2\mu_1(n-\mu_1)(\mu_1+\alpha+1)(n+\beta+1-\mu_1)}{5\epsilon(\mu_2-\mu_1)S_2}=\frac{T(\mu_1)}{10\epsilon V_1V_2}, \tag{45}$$

and

$$x_1>\mu_1+\min_{0<\epsilon<\mu_2-\hat{x}}\left(\epsilon+\sqrt{\frac{T(\mu_1)}{10\epsilon V_1V_2}}\right)=\mu_1+\frac{3}{2}\left(\frac{T(\mu_1)}{5V_1V_2}\right)^{1/3}.$$

Similarly to (42) one has

$$T(\mu_1) = ((\alpha-\beta)V_2 + U_1V_1)^2\,((\alpha-\beta)V_2 + U_2V_1)^2\,S_2^{-4}, \tag{46}$$

giving (37). □

7. Hahn polynomials $\alpha \le \beta < -n-1$

There are some marginal differences between this and the previous case related to the sign of few expressions. In particular, for the absence of zeros in $[0, 1]$ and $[n-1, n]$ now it is required $(1+\alpha)n < r$ and $(1+\beta)n < r$, respectively. Then the existence of y_0 and y_k under the imposed conditions follows from Lemma 4. The only proof we have to alter is that of (37). To make the formulas more transparent we set

$$\hat{\alpha} = -n-\alpha, \qquad \hat{\beta} = -n-\beta, \quad \hat{\alpha} \ge \hat{\beta} \ge 1.$$

It is worth noticing that $k \le n-1$ since $(1+\alpha)n - r = \hat{\beta}n > 0$ if $k = n$, i.e. $x_1 \in [0, 1]$. We need the following lemma.

Lemma 11. *Suppose that $(1+\alpha)n < r$, $\hat{\alpha} \ge \hat{\beta} \ge 1$, and if $\hat{\beta} = 1$, then $\hat{\alpha} > 1$. Then $S_2 > 0$ and the functions V_1 and V_2 are real.*

Proof. We have

$$\begin{aligned}
&S_2 + 4(1+\alpha)n - 4r > S_2 = 4(\hat{\beta}-1)n + (\hat{\alpha}+\hat{\beta}-2)^2 > 0,\\
&V_1^2 > V_1^2 + (1+\alpha)n - r = (n+\hat{\alpha}-1)(\hat{\beta}-1) \ge 0,\\
&-\frac{V_2^2}{r} > -\frac{V_2^2}{r} + (1+\alpha)n - r = (\hat{\beta}-1)n \ge 0,
\end{aligned}$$

proving the claim. □

Theorem 12. *Suppose that the conditions of Lemma 11 hold. Then*

$$x_1 > \frac{S_1 - 2V_1V_2}{S_2} + \frac{3}{2S_2}\left(\frac{((\alpha-\beta)V_2 + U_1V_1)^2\,((\alpha-\beta)V_2 + U_2V_1)^2}{4V_1V_2S_2}\right)^{1/3}, \tag{47}$$

provided $k(k+\alpha+\beta+1) > n(1+\alpha)$ and $n \ge 3$.

$$x_k < \frac{S_1 + 2V_1V_2}{S_2} - \frac{3}{2S_2}\left(\frac{((\alpha-\beta)V_2 - U_1V_1)^2\,((\alpha-\beta)V_2 - U_2V_1)^2}{V_1V_2S_2}\right)^{1/3}, \tag{48}$$

provided $k(k+\alpha+\beta+1) > n(1+\beta)$.

Proof. Setting $x = \mu_1 + \epsilon$, $\;0 < \epsilon < \hat{x} - \mu_1$, with

$$\hat{x} = \frac{(n-k+\hat{\alpha})(n-k+1)}{2n-2k+\hat{\alpha}+\hat{\beta}}$$

in our notation, we have

$$\frac{B(x)}{\Delta(x)} = \frac{2x(n-x)(n+\hat{\alpha}-1-x)(x+\hat{\beta}-1)}{\epsilon(\mu_2-x)S_2} := \frac{2T(x)}{\epsilon(\mu_2-x)S_2}.$$

Notice that $n \geq k \geq 2$ and $x > \mu_1 > 1$, by the assumption $(1+\alpha)n - r > 0$. Since

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{(n-x)(x+\hat{\beta}-1)}{\mu_2 - x} = \frac{(\mu_2 - x)^2 + (n-\mu_2)(\mu_2+\hat{\beta}-1)}{(\mu_2-x)^2} > 0,$$

then

$$\frac{B(x)}{\Delta(x)} > \frac{2x(n+\hat{\alpha}-1-x)(n-\mu_1)(\mu_1+\hat{\beta}-1)}{\epsilon(\mu_2-x)S_2}. \tag{49}$$

We claim that

$$x_1 < \hat{x} \leq \frac{n+\hat{\alpha}-1}{2},$$

hence the absolute minimum of the right hand side of (49) is attained for $x = \mu_1$. Indeed,

$$\frac{n+\hat{\alpha}-1}{2} - \hat{x} = \frac{2(n-k)(k-2) + \hat{\alpha}(n+\hat{\alpha}+\hat{\beta}-3) + \hat{\beta}(n-1)}{2(2n-2k+\hat{\alpha}+\hat{\beta})} > 0.$$

Therefore,

$$\frac{B(x)}{\Delta(x)} > \frac{2T(\mu_1)}{\epsilon V_1 V_1},$$

implying

$$x_1 > \mu_1 + 3\left(\frac{T(\mu_1)}{4V_1V_2}\right)^{1/3}.$$

Now the result follows by (46). □

8. Mesh of the symmetric Hahn polynomials $\alpha = \beta$

We will establish the following:

Theorem 13. *Let*

$$M_1(n,k,\alpha) = \sqrt{\frac{2n(n+2\alpha+2)}{r}},$$

$$M_2(n,k,\alpha) = \frac{2\sqrt{h^2 + h\sqrt{((\alpha+1)n-r)\left((\alpha+1)^2+h+r\right)}}}{(\alpha+1)^2+r},$$

where $h = (\alpha+1)(n+\alpha+1)$ *and* $r = k(k+2\alpha+1)$.

Then for $\alpha > -1$,

$$\mathcal{M}\left(Q_k^n(x;\alpha,\alpha)\right) > \begin{cases} M_1(n,k,\alpha), & r < \dfrac{(\alpha+1)^2(n+2\alpha+2)n}{(\alpha+1)^2+(n+\alpha+1)^2}, \\ M_2(n,k,\alpha), & \dfrac{(\alpha+1)^2(n+2\alpha+2)n}{(\alpha+1)^2+(n+\alpha+1)^2} \leq r < (1+\alpha)n; \end{cases} \tag{50}$$

and for $\alpha < -n-1$,

$$\mathcal{M}\left(Q_k^n(x;\alpha,\alpha)\right) > M_1(n,k,\alpha), \quad r > (1+\alpha)n. \tag{51}$$

Proof. It will be convenient to use $y = n - 2x$ and $\alpha_1 = \alpha + 1$ rather then x and α. In view of the symmetry of $Q_k^n(x; \alpha, \alpha)$ with respect to $n/2$ we may assume that $0 \le y \le n$.

We start with the case $\alpha > -1$. Since

$$\mathcal{M}^2(Q_k) > 4 \min_x \frac{\sqrt{B(x)}}{\sqrt{B(x)} - A(x)} = \min_x \frac{4}{1 - \frac{A(x)}{\sqrt{B(x)}}},$$

by (8), we can instead minimize the function

$$g(y) = \frac{A^2(x)}{B(x)} = \frac{(n^2 + 2\alpha_1 n - 2r - y^2)^2}{(n^2 - y^2)\left((n + 2\alpha_1)^2 - y^2\right)}.$$

We calculate

$$\frac{(n-y)^2(n+y)(n+2\alpha_1+y)^2(n+2\alpha_1-y)}{8A(x)} g'(y)$$
$$= y\left((\alpha_1^2 + r)y^2 - 2\alpha_1^2 r + n(n+2\alpha_1)(\alpha_1^2 - r)\right).$$

Therefore, if $r < \frac{\alpha_1^2 n(n+2\alpha_1)}{2\alpha_1^2 + 2\alpha_1 n + n^2}$ the only minimum of g is attained at $y = 0$, that is $x = \frac{n}{2}$, yielding

$$\mathcal{M}^2(Q_k) > \frac{2n(n+2\alpha_1)}{r}.$$

Otherwise the minimum is at

$$y = \sqrt{\frac{2\alpha_1^2 r - n(n+2\alpha_1)(\alpha_1^2 - r)}{\alpha_1^2 + r}},$$

giving the second case of (50).

The proof of (51) is similar with the only difference that the minimum is ever attained at $y = 0$. We omit the details. □

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